The Epsilon- Delta (Formal) Definition of a Limit... explained in English

What follows is an attempt to explain the formal definition of a limit (found in your calculus text) in more "user friendly" terms.

Epsilon = \( \varepsilon \), Delta = \( \delta \)

If you are in a hurry, here is a "quick and dirty" example problem...

\[
\lim_{x \to 1} (3x+2)=5
\]

Show that for all \( \varepsilon > 0 \), there exist \( \delta > 0 \) such that \( |(3x+2) - 5| < \varepsilon \) when \( 0 < |x - 1| < \delta \). (In other words, find a way to make \( |(3x+2)-5| \) look like \( |x-1| \).

First evaluate what's inside \( (3x+2)-5) \): \( |3x-3| < \varepsilon \)

Factor out the three: \( |3(x-1)| < \varepsilon \)

Divide by \( 3 \): \( |x-1| < \varepsilon / 3 \)

Since the absolute value of three is 3, you can take away the absolute value sign. \( |x-1| < \varepsilon /3 \)

Remember that \( 0 < |x-1| < \delta \) from way above? Since \( 0 < |x-1| < \delta \) and we just proved that \( |x-1| < \varepsilon /3 \), we can choose \( \delta = \varepsilon /3 \). This will give us \( 0 < |x-1| < \varepsilon /3 \)

If you need to know more, here is a more lengthy explanation of how it all works.....

The definition of a limit basically says that if \( L \) is a limit, you can choose \( x \) values that will make \( f(x) \) get as close to the limit \( L \) as someone demands. So, if someone challenges you to get \( f(x) \) within .0001 of the limit, you will be able to find such an \( x \) value.

In mathematical terms, if someone challenges you to get \( f(x) \) within .0001 of a limit \( L \), that is the same as saying

\[ |f(x) - L| < .0001 \]

The absolute value just indicates that it doesn't matter whether \( f(x) \) is within \( L - .0001 \) or \( L + .0001 \). \( f(x) \) just has to be within .0001 of the limit in either direction. Written more generally,

\[ |f(x) - L| < \varepsilon \]

where \( \varepsilon \) is any difference someone chooses; e.g. within .0001 of the limit, or within .000002 of the limit, or within .000000009 of the limit. In other words, no matter how small a number someone chooses, if \( L \) is a limit you can find an \( x \) such that \( f(x) \) is within that distance of the limit.

The other part of the definition of a limit is:

\[ 0 < |x - a| < \delta \]

That says that given:
1) \( L \) is a limit

2) and someone has challenged you to get \( f(x) \) within .0001 of the limit.

You can find values for \( x \) that are close to \( 'a' \) that will result in \( f(x) \) being closer than .0001 to the limit. In fact, you can state a number \( d \), e.g. .007, and all values for \( x \) within .007 of \( 'a' \) will result in \( f(x) \) being within .0001 of the limit.

For example, suppose someone says prove that: \( \lim_{{x \to 3}} (3x + 1) = 10 \)

You can interpret that to mean, "If \( f(x) \) is required to be closer than some number \( \varepsilon \) to the limit 10, show that there are values of \( x \) near \( 'a' \) that make that true." The values you are given for the problem are:

\( a = 3 \)
\( L = 10 \)
\( f(x) = 3x + 1 \)

So, plugging those values into: \( |f(x) - L| < \varepsilon \) gives: \( |3x + 1 - 10| < \varepsilon \) .......(1) and: \( |3x - 9| < \varepsilon \)

or: \( 3|x-3| < \varepsilon = |x-3| < \varepsilon /3 \)

To recap, starting at line (1) the math says that when \( |f(x) - L| \) is smaller than \( \varepsilon \), then \( |x-3| < \varepsilon /3 \). Written on the same line, it looks like this:

if \( |(3x+1) - 10| < \varepsilon \) then \( |x - 3| < \varepsilon /3 \)

Compare that line to the original definition of a limit:

if \( |f(x) - L| < \varepsilon \) then \( 0 < |x - a| < \delta \)

The two lines are identical given that \( f(x) = 3x + 1, L = 10, \) and \( a = 3 \), AND if you choose \( \delta = \varepsilon /3 \). (Well, they're identical as long as you declare the restriction \( x < a \) for the top line). So, if someone challenges you to get \( f(x) \) within some number \( \varepsilon \) of the limit, the result says you can take the number and divide it by 3, and that will give you \( x's \) that meet the challenge. For example, if someone says I want you to give me an \( x \) such that \( f(x) \) is within .001 of the limit 10. Then, you can reply that all \( x's \) within \( .001/3 = .0003333 \) of \( 'a' \) will satisfy those requirements. Note that it doesn't matter what number the person gives you, e.g. if they demand that \( f(x) \) be within \(.000007 \) of the limit, all you have to do is divide the number by 3, and any \( x \) closer to \( 'a' \) than that will satisfy their demands.

Since \( \varepsilon /3 \) is a real number for any given real number \( \varepsilon \), you have proved that there is always some range of \( x's \) such that \( f(x) \) is within \( \varepsilon \) of the limit.

The methodology for doing the proofs is to start with: \( |f(x) - L| < \varepsilon \)

and make the substitutions for \( f(x) \) and \( L \). Then you need to manipulate the inequality so that you end up with: \( |x-a| < \text{something} \)

where \( 'a' \) is one of the given. That way you can choose \( \delta \) to equal whatever is on the right side.