Finding Domain and Asymptotes of Rational Functions

A rational function is a function that is a fraction of the form \( f(x) = \frac{p(x)}{q(x)} \) where \( p(x) \) and \( q(x) \) are polynomials and \( q(x) \neq 0 \).

I. Finding Domain

In general, the domain of a rational function of \( x \) excludes all values that make the denominator zero.

1. Determine whether or not the function has a denominator with an independent variable, namely \( x \).

   Ex. \( k(x) = \frac{x^2+4x+4}{3} \) the denominator does not contain a variable; therefore the domain of \( k \) is all real numbers.

   \[ \text{Interval notation: } (-\infty, \infty) \quad \text{Set notation: } \{x|x \in \text{Real numbers}\} \]

2. If so, set the denominator equal to zero and solve for \( x \). Exclude any solutions that are real numbers from the domain.

   Ex. \( f(x) = \frac{x-1}{x^2-x-2} = \frac{x-1}{(x+1)(x-2)} \) \( (x+1)(x-2) = 0 \) \( x = -1 \quad x = 2 \)

   The domain of function \( f \) is all real numbers except \( x = -1 \) and \( x = 2 \).

   \[ \text{Interval notation: } (-\infty, -1) \cup (-1,2) \cup (2, \infty) \quad \text{Set notation: } \{x|x \neq -1, x \neq 2\} \]

3. If the denominator or the solutions of the denominator when set equal to zero are not real numbers, the domain of the function is all real numbers.

   Ex. \( h(x) = \frac{x}{x^2+1} \rightarrow x^2 + 1 = 0 \rightarrow x^2 = -1 \rightarrow x^2 = \pm\sqrt{-1} \) (no real solutions)

   The domain of \( h \) is all real numbers. Interval notation: \((-\infty, \infty)\) Set notation: \( \{x|x \in \text{Real numbers}\} \)

II. Finding Vertical Asymptotes

Vertical asymptotes (V.A.) of a function are vertical lines the graph of the function is always approaching but never touching. They serve as boundaries of the function’s graph.

1. Factor the top and bottom of the function.

2. Simplify. Any variables that cancel on top and bottom create a hole in the graph.

3. Set the denominator of the simplified function equal to zero and solve to obtain the equation(s) of vertical asymptote(s).

<table>
<thead>
<tr>
<th>Example A</th>
<th>Example B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor top and bottom ( g(x) = \frac{x}{x^2+3x} = \frac{x}{x(x+3)} ) ( x ) cancels on top and bottom leaving ( \frac{1}{x+3} ) so there is a hole in the graph at ( x=0 ). Set the new denominator equal to zero. ( x + 3 = 0 ) ( x = -3 ) is the vertical asymptote.</td>
<td>( f(x) = \frac{1}{x+2} ) Function does not factor or simplify. Set bottom equal to zero. ( x + 2 = 0 ) ( x = -2 ) is the vertical asymptote.</td>
</tr>
</tbody>
</table>
III. Finding the Horizontal Asymptote

Horizontal asymptotes (H.A.) are horizontal lines that the graph of the function approaches. Like vertical asymptotes, they serve as boundaries of the function’s graph but unlike vertical asymptotes, it is possible for a graph to go through a horizontal asymptote. To find if a function crosses its H.A., set the function equal to the H.A. and solve for $x$. Any $x$ values obtained are where the graph crosses the horizontal asymptote. If an inconsistent equation is obtained, the graph does not cross the horizontal asymptote.

When finding a H.A. we compare the degree of the numerator and denominator. There are three possibilities:

1. If the numerator has a lower degree than the denominator then the H.A. is at $y=0$, the x-axis.
   
   Ex. $f(x) = \frac{x-1}{x^2+2}$ H.A. is $y = 0$
   
   $0 = \frac{x-1}{x^2+2} \rightarrow 0 = x - 1 \rightarrow x = 1$ so the graph crosses at (1,0).

2. If the numerator and denominator have the same degree the H.A. is the ratio of the leading coefficients.
   
   Write the numerator and denominator in descending order, i.e. $f(x) = \frac{ax^n+...}{bx^n+...}$ where $n$ is the largest exponent. The H.A. is $y = \frac{a}{b}$.
   
   Ex. $g(x) = \frac{3x^2+x+1}{4x^2+2}$ H.A is $y = \frac{3}{4}$
   
   $\frac{3}{4} = \frac{3x^2+x+1}{4x^2+2} \rightarrow 4(3x^2 + x + 1) = 3(4x^2 + 2) \rightarrow 12x^2 + 4x + 4 = 12x^2 + 6 \rightarrow 4x + 4 = 6 \rightarrow 4x = 2 \rightarrow x = \frac{1}{2}$ so the graph crosses at $\left(\frac{1}{2}, \frac{3}{4}\right)$.

3. If the degree of the numerator is higher than the degree of the denominator there is no H.A. (possible oblique asymptote, see next section).
   
   Ex. $h(x) = \frac{x^2+2}{x}$ no H.A.

IV. Finding Slant/Oblique Asymptotes

Slant (also called oblique) asymptotes of a function are straight, diagonal lines that the graph of the function approaches. They serve as boundaries for the function’s graph. These asymptotes exist if and only if the degree of the numerator is exactly one greater than the degree of the denominator.

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Verify that the degree of the denominator is one less than the degree of the numerator. If not, there is not slant asymptote. $f(x) = \frac{x^3}{x^2+1}$</td>
</tr>
<tr>
<td>2.</td>
<td>Divide the numerator by the denominator using long division and drop the remainder. The result is the slant asymptote. When divide using long division we obtain $x - \frac{x}{x^2+1}$. When we drop the remainder, the slant asymptote is $y = x$.</td>
</tr>
<tr>
<td>3.</td>
<td>Set the remainder equal to zero to find if the graph will cross the slant asymptote. $\frac{-x}{x^2+1} = 0 \rightarrow -x = 0 \rightarrow x = 0$ so the graph crosses at (0,0).</td>
</tr>
</tbody>
</table>